An Approximate Method to Solve Cauchy-Type Singular Integral Equations

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Abstract

In this paper we are concerned with an approximate method to solve Cauchy-type singular integral equations (**CSIEs**). The mechanism of this method is based on an approximation of the unknown function which can be done by appropriate finite series. Galerkin method has been used to build a system of linear equations which leads to determine the unknown coefficients. Two examples are also given to show the simplicity and efficiency of the work.

Keyword: Singular integral equation, Cauchy principal value, Galarkin method.

1.Introduction

We conceder the Cauchy-type singular integral equations,

$$\int_a^b \frac{g(t)}{t-x}\,dt + \int_a^b k(x,t)\,g(t)\,dt = f(x), \qquad a<x<b \tag{1}$$

Where $a\,\&\,b$ are real constants, $f(x)$ is continuous for $[a,b]$, $k(x,t)$ is also continuous with respect to the two variables x,t on $[a,b]$, and $g(x)$ is unknown function to be determined, occurs in many problems of engineering, mathematical physics and especially in connection with the solution of partial differential equations in $\Re^2$ or $\Re^3$ [1], [2], [3]. Particularly practical problems in elasticity, crack theory, wing theory and fluid flow give rise to singular integral equations with Cauchy kernels [4]. Many methods have been proposed by different papers, the numerical evaluation of equation (1) was considered by [3], [5] and an approximate solution proposed by Z.K. Eshkuvvatov [6] , A.S. Cvetkovic [7] & G. E. Okecha [8]. The paper is organized in 5 sections. Section 2, the solution of **CSIEs** is constructed as an approximation form. In section 3, Galerkin method has been used to build a system of linear equations which leads to determine the unknown coefficients of the approximate solution. Given numerical examples in section 4, the last section contain conclusion.

2. Approximate Solution

By adding and subtracting $g(x)$ in equation (1), as the follows: [9] and [10]

$$\int_a^b \frac{g(t)-g(x)}{t-x}\,dt + g(x)\ln\left(\frac{b-x}{x-a}\right) + \int_a^b k(x,t)\,g(t)\,dt = f(x), \tag{2}$$

The technique of this method is based on the approximation of the unknown function $g(t)$, by appropriate finite series as the following: [11]

$$g(t) \approx g_n(t) = \sum_{i=0}^{n} c_i\,t^i \tag{3}$$

where $c_0, c_1, \ldots, c_n$ are unknown coefficients. Now by substituting (3) in equation (2), we will get the following:

$$\sum_{i=0}^{n} c_i \left(\int_a^b \frac{t^i - x^i}{t - x}\, dt + x^i \ln\left(\frac{b-x}{x-a}\right) + \int_a^b k(x,t)\, t^i\, dt \right) - f(x) = \mathrm{E}(x, c_0, c_1, \ldots, c_n) \qquad (4)$$

then the equation (4) will cause an error $\mathrm{E}(x, c_0, c_1, \ldots, c_n)$, this error will depend on x and on the way the coefficients $c_0, c_1, \ldots, c_n$ are chosen [12]. And since $t^m - x^m = (t-x)(t^{m-1} + x t^{m-2} + x^2 t^{m-3} + \ldots + x^{m-1})$ then

$$\frac{t^m - x^m}{t - x} = \frac{(t-x)\sum_{k=1}^{m} t^{m-k} x^{k-1}}{t-x} = \sum_{k=1}^{m} t^{m-k} x^{k-1},$$ so the equation (4) can be written as the follows:

$$\sum_{i=0}^{n} c_i \left(\int_a^b \left(\sum_{j=1}^{i} x^{i-j} t^{j-1} \right) dt + x^i \ln\left(\frac{b-x}{x-a}\right) + \int_a^b k(x,t)\, t^i\, dt \right) - f(x) = \mathrm{E}(x, c_0, c_1, \ldots, c_n) \qquad (5)$$

3. Galerkin Method [12]

By Galerkin method the error $\mathrm{E}(x, c_0, c_1, \ldots, c_n)$ is orthogonal to $n+1$ given linearly independent functions, $1, x, x^2, \ldots, x^n$ as:

$$\int_a^b x^p\, \mathrm{E}(x, c_0, c_1, \ldots, c_n)\, dx = 0, \quad p = 0, 1, \ldots, n \qquad (6)$$

then the equation (5) becomes

$$\sum_{i=0}^{n} c_i \int_a^b x^p \left(\int_a^b \left(\sum_{j=1}^{i} x^{i-j} t^{j-1} \right) dt + x^i \ln\left(\frac{b-x}{x-a}\right) + \int_a^b k(x,t)\, t^i\, dt \right) dx = \int_a^b x^p f(x)\, dx,$$
$$p = 0, 1, \ldots, n \qquad (7)$$

The equation (7) is a system of $(n+1)$ linear algebraic equations with $(n+1)$ unknown coefficients, the resulting system may be representing as a matrix form:

$$\mathrm{A}C = F \qquad (8)$$

where A is $(n+1)\times(n+1)$ matrix, whose $(p,q)^{th}$ element is:

$$\int_a^b x^p \left(\int_a^b \left(\sum_{j=1}^{q} x^{q-j} t^{j-1} \right) dt + x^q \ln\left(\frac{b-x}{x-a}\right) + \int_a^b k(x,t)\, t^q\, dt \right) dx$$

F is the $(n+1)$ vector whose p^{th} component is $\int_a^b x^p f(x)\, dx$, and C is the $(n+1)$ vector whose i^{th} component is unknown coefficient c_i.

4. Numerical Test

Example (1): Consider

$$\ln\left(\frac{1-\mathrm{x}}{\mathrm{x}}\right) + 1 = \int_0^1 \frac{\mathrm{g(t)}}{\mathrm{t-x}}\,\mathrm{dt} + \int_0^1 \mathrm{g(t)}\,\mathrm{dt}, \quad 0 < \mathrm{x} < 1,$$

the theoretical solution is $g(x)=1$, approximate solution obtained by proposed method for $n=1$ as the follows:

$$A=\begin{pmatrix}1 & 9\times10^{-8}\\ 9\times10^{-8} & 0.25\end{pmatrix} \text{ and } F=\begin{pmatrix}1\\ 9\times10^{-8}\end{pmatrix} \text{ then } C=\begin{pmatrix}c_0\\ c_1\end{pmatrix}=A^{-1}F=\begin{pmatrix}1\\ 0\end{pmatrix}$$

there for $g_1(t)=c_0+c_1 t=1$, it is coincide to the exact solution.

Example (2) [5]: Consider the problem:

$$\frac{-x\pi\sqrt{1+\alpha^2}}{|\alpha|\,(x^2+\alpha^2)}=\int_{-1}^{1}\frac{g(t)}{t-x}\,dt\ , \qquad -1<x<1\ \&\ \alpha=0.95$$

where the exact solution is: $g(x)=\dfrac{\sqrt{1-x^2}}{x^2+\alpha^2}$

by using proposed method we obtain the following approximate solutions and numerical results which is showed in the following table for different n: ($n=10$ and $n=15$) as:

$$g_{10}(x)=\sum_{n=0}^{10}c_n x^n=1.11001-0.41204x^2-10.92544x^4+39.20109x^6$$
$$-51.84048x^8+23.32813x^{10}$$

$$g_{15}(x)=\sum_{n=0}^{15}c_n x^n=1.10911-1.78514x^2+1.96117x^4-3.29368x^6+7.14030x^8$$
$$-11.96655x^{10}+10.83223x^{12}-3.94467x^{14}$$

Table: Numerical results for example2 where n=10 & n=15

x		$g_{10}(x)$	$g_{15}(x)$
0.9	0.2545342449	0.2599566253	0.2568837095
0.7	0.5128494383	0.5674713715	0.5143918777
0.5	0.7514320206	0.7569843750	0.7524515991
03	0.9611478100	1.0097415561	0.9622361333
0.1	1.0903971913	1.1048261849	1.0913428762
0.0	1.1080332410	1.1100000000	1.1090000000
-0.2	1.0395712436	1.0784185424	1.0405438807
-0.4	0.8626024838	0.8934395517	0.8637049565
-0.6	0.6336633663	0.6451044170	0.6349128856
-0.8	0.3889789303	0.4552630966	0.3901360828
-1	0.0000000000	0.4620000000	0.0510000000
L.S.E		$1.4*10^{-3}$	$1.89*10^{-6}$

5. Conclusion

The approximate technique for the solution of Cauchy-type singular integral equations presented in this paper, and further it is clear from the results in the section (4) we obtained approximate function which has rapidly convergent and some times getting completely exact solution of the aforementioned problem. In this work is extended the problem (1) in boundaries which is computed from a to b instead from -1 to 1, see [1], [2], [3], [4], [6] and [7], and the weight function in equation(1) is a unit weight function that is $w(t)=1$.

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